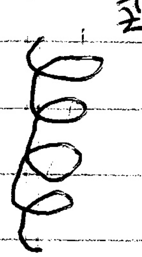
h.) cnf. homogeneous cylindrical helix.



$h \equiv$ pitch (vert. dist. in / rotation)

Invariance:

$\delta\phi$ rotation $+ \frac{\delta\phi}{2\pi} h$ vert. translation

i.e. $$\delta L = \frac{\partial L}{\partial z}\delta z + \frac{\partial L}{\partial \phi}\delta\phi$$

$$\delta z = \frac{h}{2\pi}\delta\phi, \quad \partial L/\partial z = \frac{d}{dt} p_z$$

$$\Rightarrow \quad \delta L = \frac{h}{2\pi}\dot{p}_z\,\delta\phi + \dot{L}_z\,\delta\phi$$

$$= \delta\phi\left(\frac{h}{2\pi}\dot{p}_z + \dot{L}_z\right)$$

$$\delta L = 0 \quad \Rightarrow \quad \frac{h}{2\pi} p_z + L_z = \text{const.}$$

Virial Thms

Now.... can also consider scale symmetry $\Rightarrow$ Virial Theorems.

N.B. Virial Thms are simple, interesting and useful $\rightarrow$ esp in astrophysics.

Re: scale symmetry $\Rightarrow$

if $U(\alpha \underline{r}_1, \alpha \underline{r}_2, \ldots \alpha \underline{r}_n) = \alpha^k U(\underline{r}_1, \underline{r}_2, \ldots \underline{r}_n)$

i.e. re-scale variables

$\Rightarrow$ U is homogeneous. (related to scale invariance)

Many relevant U are homogeneous, i.e.

harmonic oscillator: $k=2$

Coulomb/Gravity: $k=-1$

etc. $\Rightarrow$ homogeneous U $\Rightarrow$ power law structure.

Now, recall:

EOM from $\delta \int dt\, L = 0$, so if

$L \to \underset{\substack{\downarrow \\ \text{const. factor}}}{\sigma} L$, EOM unchanged.

$\Rightarrow$ Multiplying Lagrangian by constant factor leaves physics unchanged.

→ So, for homogeneous U, can generate class of rescalings which multiply Lagrangian by const. factor

⇒ same EOM.

→ such rescalings define basic class of relations between quantities.

→ Useful for basic story/characteristics w/o detailed crank.

Now: $$S = \int dt \left(\frac{1}{2} m \dot{\underline{r}}^2 - U(\underline{r}) \right)$$

$$r \to \alpha r'$$

$$t \to \beta t'$$

$$S = \int dt \left(\frac{1}{2} m \frac{\alpha^2}{\beta^2} \dot{r}'^2 - \alpha^k U(r') \right)$$

so if $\alpha^2/\beta^2 \sim \alpha^k$ ⇒ L multiplied by factor and δS invariant.

$\Rightarrow$ $\boxed{\beta \sim \alpha^{1-k/2} \text{ defines time-space rescaling leaving EOM unchanged.}}$

i.e. $\boxed{t'/t \sim (\ell'/\ell)^{1-k/2}}$

$\Rightarrow$ works only for homog. fctn.

Equivalently:

$$\begin{cases} v'/v \sim \alpha^{k/2} \sim (\ell'/\ell)^{k/2} \\ E'/E \sim (\ell'/\ell)^{k} \\ L'/L \sim (\ell'/\ell)^{1+k/2} \end{cases}$$

$\rightarrow$ Eg. (a) $U \sim z$ gravity $k \sim 1$

$\Rightarrow$ $t'/t \sim (\ell'/\ell)^{1/2}$

fall time ~ square of amplitude.

ⓑ $U \sim r^2$ $\quad k = 2$ (h.o.)

$t'/t \sim \ell^0$ $\Rightarrow$ period indep. of amplitude.

ⓒ $U \sim r^{-1}$ $\quad k = -1$

$$t'/t \sim (\ell'/\ell)^{3/2}$$

$$\text{Period} \sim (\text{radius})^{3/2}$$ $\Rightarrow$ Kepler's 3rd Law.

Homogeneous $U \Leftrightarrow$ Virial Thms!

What is a Virial Thm?

- consider a system of particles

- $$\frac{d}{dt} \sum_i \underline{p}_i \cdot \underline{x}_i = \sum_i \underline{p}_i \cdot \dot{\underline{x}}_i + \sum_i \dot{\underline{p}}_i \cdot \underline{x}_i$$

$$= \underset{\substack{\downarrow \\ KE}}{2T} - \sum_i \frac{\partial U}{\partial \underline{x}_i} \cdot \underline{x}_i$$

Now consider $\left\langle \left(\sum_i \underline{p}_i \cdot \underline{x}_i \right)^{\cdot} \right\rangle$ → time avg.

$$\langle A \rangle = \frac{1}{T} \int_0^T A \quad \text{as } T \to \infty$$

So if $\sum_i \underline{p}_i \cdot \underline{x}_i$ bounded in time

$$\left\langle \frac{d}{dt} \sum_i \underline{p}_i \cdot \underline{x}_i \right\rangle = \lim_{T\to\infty} \frac{1}{T} \int_0^T dt \left(\frac{d}{dt} \sum_i \underline{p}_i \cdot \underline{x}_i \right)$$

$$\to 0$$

so

$$2\langle T \rangle = \left\langle \sum_i \frac{\partial U}{\partial \underline{x}_i} \cdot \underline{x}_i \right\rangle$$

Now, if $U = U(\underline{x}_1, \underline{x}_2, \dots x_n)$
and U homogeneous

i.e. $U(\alpha \underline{x}_1, \dots \alpha x_n) = \alpha^k U(\underline{x}_1, \dots x_n)$

then $\boxed{2\langle T \rangle = K \langle U \rangle}$.

and also, of course, have:

$$T + U = E = \langle T \rangle + \langle U \rangle$$

so

$$2\langle T \rangle = k \langle U \rangle$$

$$E = \langle T \rangle + \langle U \rangle$$

$$\Rightarrow \boxed{\begin{aligned} \langle U \rangle &= \frac{2}{k+2} E \\ \langle T \rangle &= kE/k+2 \end{aligned}}$$

check:

$\rightarrow$ $k = 2$ $\qquad$ $\langle U \rangle = E/2$

$\langle T \rangle = E/2$

equipartition in H.O.

$\rightarrow$ $k = -1$

$$\langle U \rangle = E$$

$$\langle T \rangle = -E$$

$\rightsquigarrow \langle T \rangle = -E$ $\Rightarrow$ total energy negative for gravitationally bound cluster

i.e. must have boundstate for time avg. to converge.

Who cares / why care?

- virial thms relate energies to potential structure
- can relate measured K.E. (Doppler spectroscopy) to energies
- Virial is single # characterizing a cluster

$$f(\underline{x}, \underline{v}, t) \rightarrow V(\underline{x}, t) \rightarrow \langle T \rangle, \langle U \rangle, E.$$

Boltzmann $\rightarrow$ fluid $\rightarrow$ virial

$\downarrow$ velocity moment

$\downarrow$ $\sum_i \rightarrow$ integral in space,

Constraints

i.e. $r_{ij} \to r_{ij}^2 = c_{ij}^2$

Ex ① rigid body (r_{ij} constant)
② gas in container (inside walls)
③ particle moving above spherical surface ($x^2+y^2+z^2-a^2 \geq 0$)

④ particle moving on wire ($x^2+y^2=a^2$)

(30)

⑤ "rolling without slipping" $\underline{v} + \underline{r} \times \underline{\Omega} = 0$

(i) → pt contact stationary

Types:

i.) Holonomic: Expressible in form:

$$f(\underline{r}_1, \underline{r}_2 \dots r_n, t) = 0$$

more generally: $f(q_1, q_2, \dots q_n, t) = 0$

i.e. relation coordinates, time (only)

Ex ①, ④

ii.) Nonholonomic: all others.

i.e. inequalities (Ex. ③),
velocity dependence (Ex. ②), (Ex. ⑤ (albeit convertible))

Also: schleronomic $\longleftrightarrow$ independent time
rheonomic $\longleftrightarrow$ depends on time

Consider $\begin{cases}\text{holonomic constraint:} \\ \text{or} \\ \text{nonholonomic of form } \left\{ df = \sum_k a_{\alpha k}\, dq_k + a_{\alpha t}\, dt \right\}\end{cases}$ $\Rightarrow$ i.e. can eliminate directly!

$\alpha = 1 \ldots n; \quad f_\alpha(q_1, \ldots, t) = 0$

or retain, with Lagrange multiplier

For eqns. motion, extremize: $\Rightarrow$ for force constr.

$$S' = \int dt \left(L(q_i, \dot{q}_i) + \sum_{\alpha=1}^{n} \lambda_\alpha f_\alpha(q_i) \right)$$

$\begin{cases} m \text{ eqns.} \\ n \text{ constr.} \\ m+n \text{ Var} \end{cases}$ (n add'l $\Rightarrow \lambda$'s)

$\lambda_\alpha \longleftrightarrow$ Lagrange multiplier

$\therefore$ Lagrange Eqns:

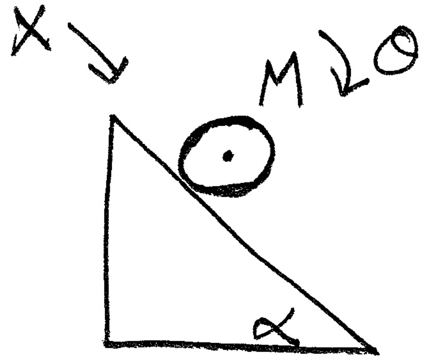
$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} - \sum_{\alpha=1}^{n} \lambda_\alpha \frac{\partial f_\alpha}{\partial q_i} = 0}$$

Note: Can write $\displaystyle \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} - \underbrace{Q_i^{con}}_{\text{gen. force}} = 0$

$$Q_i^{con} = \sum_{\alpha=1}^{n} \lambda_\alpha \frac{\partial f_\alpha}{\partial q_i}$$

Forces of constraint

Ex. 1



Cylinder rolling down incline ($I = \frac{1}{2} M R^2$)

G.C.: x, θ

$$T = \frac{1}{2} M \dot{x}^2 + \frac{1}{2} I \dot{\theta}^2$$

$$V = - M g x \sin\alpha$$

Constraint: $x - R\theta = 0$

so

$$L = \frac{1}{2} M \dot{x}^2 + \frac{1}{2} I \dot{\theta}^2 + M g \sin\alpha \, x + \lambda (x - R\theta)$$

$\Rightarrow$

$$M \ddot{x} = M g \sin\alpha + \lambda$$ force of constraint (i.e. friction $\Rightarrow$ roll without slip)

$$I \ddot{\theta} = -\lambda R$$ force of constraint.

$$x = R\theta \Rightarrow \ddot{x} = R\ddot{\theta}$$

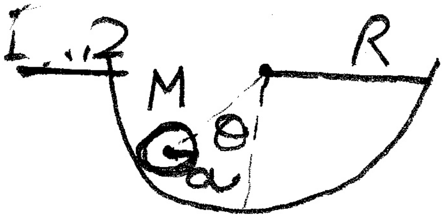
$$\Rightarrow \lambda = \frac{-I\ddot{\theta}}{R} = \frac{-I\ddot{x}}{R^2}$$

and $\ddot{x} = \frac{2}{3} g \sin\alpha$ → acceleration due constraint

$\lambda = -\frac{1}{3} Mg \sin\alpha$ → force upward (friction → allows rolling)

Note: For holonomic constraint:
→ can eliminate directly, using $f(q_1, \ldots q_n, t) = 0$
* → but, using Lagrange multipliers allows determination of force of constraint.

impt



Sphere rolling cylinder oscillation frequency, forces of constraint ?

G. C. : θ, ϕ

$$T = \tfrac{1}{2} M (R-a)^2 \dot{\theta}^2 + \tfrac{1}{2} I \dot{\phi}^2$$

$$V = Mg(R-a)(1-\cos\theta)$$

$$(R-a)\theta - a\phi = 0 \longleftrightarrow \text{Eqn. Constraint (Rolling)}$$

so
$$L = \tfrac{1}{2} M (R-a)^2 \dot{\theta}^2 + \tfrac{1}{2} I \dot{\phi}^2 - Mg(R-a)(1-\cos\theta) + \lambda\big((R-a)\theta - a\phi\big)$$

$$\Rightarrow \frac{d}{dt}\left(M(R-a)^2\dot{\theta}\right) = -Mg(R-a)\sin\theta + \lambda(R-a)$$

$$\frac{d}{dt}\left(I\dot{\phi}\right) = -\lambda a$$

$\longleftrightarrow$ constraint 'force' (torque)

$$\therefore \begin{cases} (R-a)\ddot{\theta} = -g\sin\theta + \dfrac{\lambda}{M} \\ I\ddot{\phi} = -\lambda a \\ (R-a)\ddot{\theta} - a\ddot{\phi} = 0 \end{cases}$$

{3 eqn; 3 unknown

so $$\ddot{\phi} = \frac{(R-a)}{a}\ddot{\theta}$$

$$\Rightarrow \quad X = -\frac{I(R-a)}{Ma^2}\ddot{\theta}$$

So
$$\left((R-a) + \frac{I(R-a)}{a^2}\right)\ddot{\theta} + g\,\theta = 0$$

for $\theta \ll 1$.

Note: Force of constraint varies in time

Aside: Rayleigh Dissipation Function

How include friction in Lagrangian mechanics (so as to get benefit of generalized coordinates)?

Observe: usually, simple friction has form:

$$\underline{F}_f = -k\underline{v}$$

so, can define:

$$\mathcal{F} = \frac{1}{2} k \underline{v}^2 \rightsquigarrow \text{Rayleigh Dissipn. Function}$$

$$= \frac{1}{2} k \dot{\underline{q}}^2$$

(Power exerted vs. friction)

i.e.

$$\underline{F}_f = -\frac{\partial}{\partial \dot{\underline{q}}} \mathcal{F}$$

∴ can add a generalized force:

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} + \frac{\partial \mathcal{F}}{\partial \dot{q}_i} = 0}$$

generalized Lagrange Eqns.

{ incorporates friction in Lagrangian framework.

3.) Relation of Lagrangian Trajectories to Geodesics, for free particle.

Recall for free particle;

trajectory: $\delta S = 0$

$$S = \int dt\, T$$

$$T = \frac{1}{2} m \left(\frac{d\ell}{dt}\right)^2$$ (minimizes action)

for geodesic;

path: $\delta \int d\ell = 0$ (minimizes distance).

But

$$d\ell^2 = \sum_{i,k} g_{ik}(q)\, dq^i dq^k$$

$$d\ell = \left(\sum_{i,k} g_{ik}(q)\, dq^i dq^k\right)^{1/2}$$

Energy conserved $\Rightarrow$

$$E = \frac{1}{2} m \left(\frac{d\ell}{dt}\right)^2 = \frac{1}{2} m \sum_{i,k} g_{ik} \frac{dq^i}{dt} \frac{dq^k}{dt}$$

so $$dt = \left(\frac{m}{2E}\right)^{1/2} \left(\sum_{i,k} g_{ik}\, dq^i dq^k\right)^{1/2}$$

so

$$S = \int \frac{\frac{m}{2} \sum_{jk} g_{jk}\, dq^j dq^k}{\left(\frac{m}{2E}\right)^{1/2} \left(\sum_{jk} g_{jk}\, dq^j dq^k\right)^{1/2}}$$

$$= \left(\frac{Em}{2}\right)^{1/2} \int \left[\sum_{jk} g_{jk}\, dq^j dq^k\right]^{1/2}$$

$$= \left(\frac{Em}{2}\right)^{1/2} \int d\ell$$

$\therefore$ Action is simply distance (up to constant multiplier) for free particle.

$\Rightarrow$ Natural correspondence between free particle trajectories and geodesic curves.